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Old and New Inequalities

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Chapter 1

Problems

1. Prove that for all positive real numbers a, b the following inequality holds

$$\sqrt{2a(a+b)^3} + b\sqrt{2(a^2+b^2)} \leq 3(a^2+b^2).$$

2. Consider real numbers a, b, c contained in the interval $[\frac{1}{2}, 1]$. Prove that

$$2 \leq \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b} \leq 3.$$

3. Let a, b, c be three positive real numbers contained in the interval $[0, 1]$. Prove that

$$\frac{1}{ab+1} + \frac{1}{bc+1} + \frac{1}{ca+1} \leq \frac{5}{a+b+c}.$$

4. Let x, y, z be positive real numbers such that $xyz = 1$. Show that the following inequality holds:

$$\frac{1}{(x+1)^2+y^2+1} + \frac{1}{(y+1)^2+z^2+1} + \frac{1}{(z+1)^2+x^2+1} \leq \frac{1}{2}.$$

5. Let a, b, c be three positive real numbers satisying $abc = 8$. Prove that

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0.$$

6. Let $a,\ b,\ c$ be the sidelengths of an acute-angled triangle. Prove that

$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3) \geq 4(a^6+b^6+c^6).$$

7. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$a\sqrt{b^2+c^2+bc} + b\sqrt{c^2+a^2+ca} + c\sqrt{a^2+b^2+ab} \geq 3.$$

8. Find the maximum value of

$$(x^3+1)(y^3+1),$$

for all real numbers x, y, satisfying the condition that $x+y=1$.

9. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{b+c}+\frac{b+c}{a+b}+1.$$

10. If x, y, z are positive real numbers, prove that the following inequality holds:

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

11. Let a, b, c be positive real numbers such that

$$a+b+c \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$a+b+c \geq \frac{3}{a+b+c}+\frac{2}{abc}.$$

12. Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geq 1.$$

Prove that $a+b+c \geq ab+bc+ca$.

13. Let a, b, c be real numbers satisfying $a,\ b,\ c \geq 1$ and $a+b+c=2abc$. Prove that

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{ab-1}+\sqrt[3]{bc-1}+\sqrt[3]{ca-1}.$$

14. Let $a_1,\ a_2,\ \ldots, a_n$ be positive real numbers satisfying the condition that $a_1+a_2+\ldots+a_n=1$. Prove that

$$\sum_{j=1}^{n} \frac{a_j}{1+a_1+\ldots+a_j} < \frac{1}{\sqrt{2}}.$$

15. Positive numbers α, β, $x_1,\ x_2,\ \ldots,\ x_n$ $(n \geq 1)$ satisfy the condition $x_1+x_2+\ldots+x_n=1$. Prove that

$$\frac{x_1^3}{\alpha x_1+\beta x_2}+\frac{x_2^3}{\alpha x_2+\beta x_3}+\ldots+\frac{x_n^3}{\alpha x_n+\beta x_1} \geq \frac{1}{n(\alpha+\beta)}.$$

16. If three nonnegative real numbers a, b, c satisfy the condition

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2,$$

prove that

$$ab+bc+ca \leq \frac{3}{2}.$$

17. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}.$$

18. If x, y, z are positive numbers satisfying the condition $xy+yz+zx=1$, show that

$$\frac{27}{4}(x+y)(y+z)(z+x)\geq\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2\geq 6\sqrt{3}.$$

19. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{(a-c)^2}{ab+bc+ca}.$$

20. Let a, b, c be nonnegative real numbers satisfying $ab+bc+ca=3$. Prove that

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\leq\frac{3}{1+2abc}.$$

21. Let a, b, c be positive real numbers such that $2a+b=1$. Prove that

$$\frac{5a^3}{bc}+\frac{4b^3}{ca}+\frac{3c^3}{ab}\geq 4.$$

22. *i)* If x, y and z are three real numbers, all different from 1, such that $xyz=1$, then prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1.$$

ii) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

23. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)^2}+\frac{b}{c(c+a)^2}+\frac{c}{a(a+b)^2}\geq\frac{9}{4(ab+bc+ca)}.$$

24. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c}\leq\sqrt{3}.$$

25. Let a, b, c be the sidelengths of a triangle. Prove that

$$\sum_{cyc}\frac{a^3}{a^3+(b+c)^3}+1\geq 2\sum_{cyc}\frac{a^2}{a^2+(b+c)^2}.$$

26. Prove that for any real numbers a, b, c the following inequality holds

$$(b+c-a)^2(c+a-b)^2(a+b-c)^2\geq(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2).$$

27. Let a, b, c be the sidelengths of a given triangle. Prove that

$$(a+b)(b+c)(c+a)+(-a+b+c)(a-b+c)(a+b-c) \geq 9abc.$$

28. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right) \geq \frac{9}{abc+1}.$$

29. Let a, b, c be positive real numbers contained in the interval $[0,1]$. Prove that

$$\frac{2a}{1+bc}+\frac{2b}{1+ca}+\frac{2c}{1+ab}+abc \leq 4.$$

30. Let a, b, c be nonnegative real numbers a, b, c satisfying

$$\max\{b+c-a,\ c+a-b,\ a+b-c\} \leq 1.$$

Prove that

$$a^2+b^2+c^2 \leq 1+2abc.$$

31. If x, y, z are real numbers satisfying $xyz=-1$, prove that

$$x^4+y^4+z^4+3(x+y+z) \geq \frac{y^2+z^2}{x}+\frac{z^2+x^2}{y}+\frac{x^2+y^2}{z}.$$

32. Let a, b, c, d be positive real numbers satisfying the condition $a+b+c+d = abc+bcd+cda+dab$. Prove that

$$a+b+c+d+\frac{2a}{a+1}+\frac{2b}{b+1}+\frac{2c}{c+1}+\frac{2d}{d+1} \geq 8.$$

33. Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a^2+2bc}{b^2+c^2}+\frac{b^2+2ca}{c^2+a^2}+\frac{c^2+2ab}{a^2+b^2} \geq 3.$$

34. Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)} \geq \frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}.$$

35. Let a, b, c be positive real numbers such that $ab+bc+ca \geq 3$. Prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}.$$

36. Let x, y, z, t be positive real numbers such that

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}+\frac{1}{t+1}=1.$$

Prove that

$$\begin{aligned}&\min\left\{\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\frac{1}{y}+\frac{1}{z}+\frac{1}{t},\frac{1}{z}+\frac{1}{t}+\frac{1}{x},\frac{1}{t}+\frac{1}{x}+\frac{1}{y}\right\}\le 1\\ \le\ &\max\left\{\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\frac{1}{y}+\frac{1}{z}+\frac{1}{t},\frac{1}{z}+\frac{1}{t}+\frac{1}{x},\frac{1}{t}+\frac{1}{x}+\frac{1}{y}\right\}.\end{aligned}$$

37. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n}a_j^{T_k}\right)\ge\left(\sum_{k=1}^{n}a_k^{\frac{T_n+1}{3}}\right)^n,$$

where $T_k=\frac{k(k+1)}{2}$ is the k-th triangular number.

38. Let a, b, c, d be positive numbers. Prove that

$$3(a^2-ab+b^2)(c^2-cd+d^2)\ge(a^2c^2-abcd+b^2d^2).$$

39. Let a, b, c be real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\le\frac{9}{10}.$$

40. Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}.$$

41. Let a, b, c be positive real numbers such that $a+b+c+1=4abc$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\ge\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}.$$

42. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Set $x=\sqrt{a^2-a+1}$, $y=\sqrt{b^2-b+1}$ and $z=\sqrt{c^2-c+1}$. Prove that: $xy+yz+zx\ge 3$ and $x+y+z\le 2+\sqrt{7}$.

43. Let $n\ge 2$ be a given integer. Determine

(a) the largest real c_n such that

$$\frac{1}{1+a_1}+\frac{1}{1+a_2}+\ldots+\frac{1}{1+a_n}\ge c_n$$

holds for any positive numbers $a_1, a_2, \ldots, a_n$ with $a_1 a_2 \ldots a_n = 1$.

(b) the largest real d_n such that

$$\frac{1}{1+2a_1} + \frac{1}{1+2a_2} + \ldots + \frac{1}{1+2a_n} \geq d_n$$

holds for any positive numbers $a_1, a_2, \ldots, a_n$ with $a_1 a_2 \ldots a_n = 1$.

44. Let a, b, c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc} + \frac{ca}{b^2+ca} + \frac{ab}{c^2+ab} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

45. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let $d = \max\{d_i \mid 1 \leq i \leq n\}$.

(a) Prove that for any real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$, we have

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}.$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$ such that we have equality in (a).

46. Let a, b, c be nonzero positive numbers. Prove that

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}} + \sqrt{\frac{b^2}{4b^2+bc+4c^2}} + \sqrt{\frac{c^2}{4c^2+ca+4a^2}} \leq 1.$$

47. Let a, b, c be positive numbers such that $4abc = a+b+c+1$. Prove that

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c} \geq 2(ab+bc+ca).$$

48. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{(a+b)^3} + \frac{b^3}{(b+c)^3} + \frac{c^3}{(c+a)^3} \geq \frac{3}{8}.$$

49. Let a, b, c, x, y, z be positive real numbers. Prove that

$$(a^2+x^2)(b^2+y^2)(c^2+z^2) \geq (ayz+bzx+cxy-xyz)^2.$$

50. Let x, y, z be positive real numbers. Prove that

$$\frac{\sqrt{y+z}}{x} + \frac{\sqrt{z+x}}{y} + \frac{\sqrt{x+y}}{z} \geq \frac{4(x+y+z)}{\sqrt{(y+z)(z+x)(x+y)}}.$$

51. Let a, b, c be nonnegative real numbers such that $abc = 4$ and $a,\ b,\ c > 1$. Prove that

$$(a-1)(b-1)(c-1)\left(\frac{a+b+c}{3}-1\right) \le \left(\sqrt[3]{4}-1\right)^4.$$

52. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \ge \frac{3}{2}.$$

53. Prove that for all positive real numbers a, b, c the following inequality holds:

$$\frac{1}{a+b+c}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \ge \frac{1}{ab+bc+ca}+\frac{1}{2(a^2+b^2+c^2)}.$$

54. Let a, b, c be the sidelengths of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 3.$$

55. Let a, b, c be the sidelengths of a triangle with perimeter 1. Prove that

$$1 < \frac{b}{\sqrt{a+b^2}}+\frac{c}{\sqrt{b+c^2}}+\frac{a}{\sqrt{c+a^2}} < 2.$$

56. Prove that for any positive real numbers a, b and c, we have that

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \ge \sqrt{6\cdot\frac{a+b+c}{\sqrt[3]{abc}}}.$$

57. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{ab+b^2}}+\frac{b}{\sqrt{bc+c^2}}+\frac{c}{\sqrt{ca+a^2}} \ge \frac{3}{\sqrt{2}}.$$

58. Let $a_1 \le a_2 \le \ldots \le a_n$ be positive real numbers such that

$$\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}=1, \quad \frac{a_1+a_2+\ldots+a_n}{n}=m,$$

where $1 \ge m > 0$. Prove that for all i satisfying $a_i \le m$, we have

$$n-i \ge n(m-a_i)^2.$$

59. Let x, y, z be positive real numbers. Prove that

$$\frac{3\sqrt{3}}{2} \le \sqrt{x+y+z}\cdot\left(\frac{\sqrt{x}}{y+z}+\frac{\sqrt{y}}{z+x}+\frac{\sqrt{z}}{x+y}\right).$$

60. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}+\frac{c}{\sqrt{c^2+2ab}}\leq\frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

61. Let a, b, c be distinct positive real numbers. Prove the following inequality:

$$\frac{a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-6abc}{a^2+b^2+c^2-ab-bc-ca}\geq\frac{16abc}{(a+b+c)^2}.$$

62. Let a, b, c be nonzero positive real numbers. Prove that

$$\frac{a^3+abc}{b+c}+\frac{b^3+abc}{c+a}+\frac{c^3+abc}{a+b}\geq\frac{a(b^3+c^3)}{a^2+bc}+\frac{b(c^3+a^3)}{b^2+ca}+\frac{c(a^3+b^3)}{c^2+ab}.$$

63. Let a, b, c, d be real numbers with sum 0. Prove the inequality:

$$(ab+ac+ad+bc+bd+cd)^2+12\geq 6(abc+abd+acd+bcd).$$

64. Let a, b, c be positive real numbers satisfying $a+b+c=1$. Prove that

$$\left(\frac{1}{a}-2\right)^2+\left(\frac{1}{b}-2\right)^2+\left(\frac{1}{c}-2\right)^2\geq\frac{8(a^2+b^2+c^2)^2}{(1-a)(1-b)(1-c)}.$$

65. Let $x_1,\ x_2,\ \ldots,\ x_n$ be real numbers from the interval $[0,\ 1]$ satisfying

$$x_1x_2\ldots x_n=(1-x_1)^2(1-x_2)^2\ldots(1-x_n)^2.$$

Find the maximum value of $x_1x_2\ldots x_n$.

66. Let a, b, c be three positive real numbers with sum 3. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq a^2+b^2+c^2.$$

67. Let $a,\ b,\ c$ be positive real numbers satisfying $a+b+c=3$. Prove that

$$\frac{a}{2b+1}+\frac{b}{2c+1}+\frac{c}{2a+1}\leq\frac{1}{abc}.$$

68. For any three positive numbers a, b, c, prove the inequality

$$(1+abc)\left(\frac{1}{a\,(1+b)}+\frac{1}{b\,(1+c)}+\frac{1}{c\,(1+a)}\right)\geq 3.$$

69. Let $a,\ b,\ c,\ d$ be real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-cd}+\frac{1}{1-da}\leq\frac{16}{3}.$$